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A Statistical Mechanical Interpretation of Black Hole Entropy Based on an Orthonormal Frame Action

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Abstract

Carlip has shown that the entropy of the three-dimensional black hole has its origin in the statistical mechanics of microscopic states living at the horizon. Beginning with a certain orthonormal frame action, and applying similar methods, I show that an analogous result extends to the (Euclidean) black hole in any spacetime dimension. However, this approach still faces many interesting challenges, both technical and conceptual.

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Introduction

General relativity predicts black holes, which are completely characterized by just two parameters: their mass and angular momentum. General relativity also provides laws of black hole mechanics involving these parameters, which are strikingly similar to the laws of thermodynamics provided one makes certain identifications, such as the black hole mass with its energy, one-quarter the horizon area with its entropy, and so on [1,2]. Hawking showed that this similarity is not accidental: the laws of black hole mechanics *are* the laws of thermodynamics as applied to a black hole [3]. But the thermodynamics of any ordinary physical system is only an approximation based on a more fundamental statistical mechanical description of its microscopic degrees of freedom. On this last point general relativity seems to be peculiarly silent. So perhaps general relativity is not a fundamental description of gravity and we must search for a deeper theory. Or—the point of view taken here—it contains a subtlety that when treated carefully will yield a classical description of certain microscopic degrees of freedom, which when quantized will yield a highly degenerate set of microstates, and ultimately a statistical mechanical explanation for black hole entropy. It is such a subtlety I will discuss in this paper.

The ideas here are inspired by two main themes which stand out in the literature. The first is that black hole entropy is intimately connected with topological considerations, as first emphasized by Gibbons and Hawking in 1979 [4]. Perhaps the strongest statement of this idea are recent arguments which suggest that the Bekenstein-Hawking entropy formula ($S = A/4$), generalized to encompass arbitrary topology, should read [5]

$$S = \frac{\chi}{8} A \,, \tag{0.1}$$

where χ is the Euler number of the black hole, and A is the horizon area in Planck units. Topological considerations will enter the analysis here in at least two places. Firstly, the analysis will be based on an orthonormal frame formulation of general relativity, and a nontrivial spacetime topology is closely linked to the necessary existence of some set of singular points at which the frame is multivalued. The action I will be using forces us to excise any such singular points, introducing boundaries on which, it turns out, the physically relevant microstates live. (The general idea of microstates living on boundaries is introduced more fully beginning in the next paragraph.) Secondly, some preliminary results I will describe indicate that a semiclassical counting of the degeneracy of the microstates should involve precisely the two parameters A and χ, in a way suggestive of the general-topology entropy formula (0.1).

The second inspirational theme is the physically plausible idea that the microscopic degrees of freedom, if they exist at all within the context of general relativity, will likely be associated in some way with the event horizon (thought of as a 'boundary' of sorts); after all, the entropy is proportional to the area of the event horizon. The first concrete realization of this idea occured in the context of the (2+1)-dimensional black hole discovered by Bañados, Teitelboim, and Zanelli (BTZ) [6], when Carlip showed that its entropy can be understood in terms of the statistical mechanics of certain microstates living at a boundary defined by apparent horizon boundary conditions [7]. Closely related is the independent

work by Balachandran, Chandar, and Momen [8]. The basic idea in Carlip's approach is to use the fact that in three dimensions general relativity can be formulated as a (Chern-Simons) gauge theory [9, 10], and the presence of a boundary breaks the gauge symmetry leading to a Wess-Zumino-Witten (WZW) boundary action [11, 12] for "would-be gauge" degrees of freedom now promoted to physical gravitational boundary degrees of freedom. Enforcing apparent horizon boundary conditions and quantizing the resulting boundary theory leads to a set of microstates whose degeneracy correctly accounts for the BTZ black hole entropy [7, 13, 14].[1] But this approach seems to rely on a gauge-theoretic formulation of general relativity, making it specific to gravity in three dimensions. I will show that this might unnecessarily be asking too much: all we require is a formulation of general relativity which has a gauge symmetry that is broken when a boundary is present. (I.e., we do not require such gauge transformations to be equivalent on-shell to diffeomorphisms, as is the case in three-dimensional gravity [10].) I will introduce such a formulation and show how it opens the door to extending the main idea of Carlip's approach to black holes in any spacetime dimension.

Baez *et al* [16] have also found evidence for boundary degrees of freedom, this work in the context of the loop variables approach to quantum gravity. More recently, Ashtekar *et al* [17] have introduced a 'black hole sector' of non-perturbative canonical quantum gravity in which the quantum black hole degrees of freedom (i.e., the microstates) are described by a Chern-Simons field theory on the horizon (see also Refs. [18, 19]). They show that, for the case of a large non-rotating black hole, counting the degeneracy of these microstates leads to an entropy proportional to the area A divided by the so-called Immirzi parameter. An appropriate choice of this parameter then yields the Bekenstein-Hawking entropy. (A somewhat analogous undetermined parameter appears in my analysis—I will comment on this in Section 6.)

Progress towards understanding black hole microstates has also been made on the string theory front. (A review of some of the original ideas can be found in Ref. [20].) For an example immediately relevant here, Sfetsos and Skenderis [21] have shown that the ordinary (four-dimensional) Schwarzschild black hole is U-dual to the (three-dimensional) BTZ black hole so one can count its microstates by counting the BTZ black hole microstates following Carlip's approach, and they find the correct Bekenstein-Hawking entropy. We will encounter numerous other hints that the approach described here may have hidden connections with the string theory approach to black hole microstates.

The main point of the previous three paragraphs is to emphasize that the idea that microstates can be somehow associated with boundaries, in particular the event horizon, is a viable one. My point of view then is to simply do general relativity on a manifold with boundary and see what happens. Mathematically this certainly opens up a wealth of possibilities, and whether or not the results are physically meaningful—in particular, *can* we think of the event horizon as a boundary, and if so, precisely *how*—can be decided after the analysis. Nevertheless, at this point I can raise some anticipated objections. For instance, one might be uncomfortable thinking of the horizon as a 'tangible' boundary, or might wonder, "What about the microstates which would presumably reside also at the boundary

[1] For a recent review, and discussion of some unresolved questions in this approach, see Ref. [15].

at infinity?" Such concerns will be neatly dealt with in the approach I will now outline.

In Section 1 we will begin with a d-dimensional 'bare' manifold, M, with boundary ∂M, not being at all specific about the physical nature of this boundary. We will then endow this manifold with an orthonormal frame and suppose that, fundamentally, this frame (rather than the metric) encodes the gravitational degrees of freedom. This introduces additional degrees of freedom, namely local frame rotations, which we will want to be pure gauge at least in the interior of M, but not necessarily on its boundary. We will then introduce a very natural action, first order in derivatives of the frame field, which is gauge invariant except for a boundary term that breaks the gauge symmetry on ∂M. Applying the main idea of Carlip's program then leads to a certain boundary action for "would-be gauge" frame rotation degrees of freedom now promoted to physical gravitational boundary degrees of freedom. In the remainder of the paper I will discuss the nature and interpretation of this boundary theory, restricting attention mainly to the case of Euclidean black holes.

To begin with, in Section 2 I will describe where the microstates live, and how this question is related to singularities (i.e., multivaluedness) of the frame. Section 3 is devoted to the action principle, boundary conditions, and Euler-Lagrange equations. Sections 4 and 5 contain a description of the boundary theory phase space and its relation to a higher dimensional generalization of Kač-Moody and Virasoro algebras. Finally, in Section 6 I will address the issues of quantization and the counting of the degeneracy of the microstates. Unfortunately, at the technical level this is a rather formidable task, and some detailed results are worked out in the three spacetime dimensions case only. For the higher dimensional cases it is perhaps worth noting at this point that the mathematics involved is reminiscent of that used in the study of p-branes and non-perturbative effects in string theory.

1 The Proposed Action

In a metric formulation of gravity the gravitational degrees of freedom are encoded in the metric, g, and the standard action, second order in derivatives of the metric, is

$$I^{(2)}[g] = \frac{1}{2\kappa} \int_M \epsilon \, R \, . \tag{1.1}$$

Here M is the spacetime manifold, on which g induces a volume form, ϵ, and scalar curvature R. In d spacetime dimensions κ is a numerical factor times $(l_P)^{d-2}$, where l_P is the Planck length. It sets the scale of the action, and enters any quantum statement following from this action. If the manifold has a boundary, ∂M, we must consider what boundary term (if any) to add to this action, knowing that the choice we make is important to the physics we are trying to describe. For instance, one might add a boundary term to accommodate certain prescribed boundary conditions, such as microcanonical boundary conditions [22], but in any case I wish to emphasize that this boundary term is usually put in by hand.

Now, for reasons which will soon become apparent, let us switch to a formulation of gravity in which the gravitational degrees of freedom are encoded, not in the metric, but rather its 'square root': the one-form fields e^a, $a = 0, 1, \ldots, d-1$, which at any given spacetime point represent, physically, the orthonormal reference frame associated with an

observer at that point. It is a simple exercise to show that

$$\epsilon\, R = \omega^a{}_b \wedge \omega^b{}_c \wedge \epsilon^c{}_a - d\left(\omega^a{}_b \wedge \epsilon^b{}_a\right), \tag{1.2}$$

where (and this is important) the connection one-form, $\omega^a{}_b$, is not an independent field, but rather it is determined uniquely from a given frame field through the usual metricity ($\omega^{ab} = -\omega^{ba}$) and no-torsion ($de^a + \omega^a{}_b \wedge e^b = 0$) conditions. We have defined the $(d-2)$-form $\epsilon_{ab} = i_{e_b} i_{e_a} \epsilon$, where the vector fields e_a, dual to the one-form fields e^a, are defined by the relations $i_{e_a} e^b = \delta^b_a$, and frame indices are raised and lowered with the matrix $\eta^{ab} = \eta_{ab}$, diagonal with s -1's and $(d-s)$ $+1$'s, reflecting the signature of the spacetime metric. The frame rotation group is $G = SO(d-s,s)$, with Lie algebra $g = so(d-s,s)$.[2]

It is useful to introduce the notation

$$E = \text{matrix}\left(-\epsilon^a{}_b\right), \quad W = \text{matrix}\left(\omega^a{}_b\right), \tag{1.3}$$

with a (b) a row (column) index, and to think of E as a g-valued $(d-2)$-form, and W as a g-valued one-form. The gravitational action I am advocating is first order in derivatives of the frame field and comes from simply dropping the total divergence on the right hand side of (1.2):

$$I^{(1)}[e] = -\frac{1}{2\kappa}\int_M \mathrm{Tr}\left(W \wedge W \wedge E\right) = I^{(2)}[g(e)] + I_{\partial M}[e]\,, \quad I_{\partial M}[e] = -\frac{1}{2\kappa}\int_{\partial M} \mathrm{Tr}\left(W \wedge E\right), \tag{1.4}$$

where Tr denotes matrix trace. Notice that $I^{(1)}[e]$ suggests a boundary term, namely $I_{\partial M}[e]$, that is to be added to the bulk action in (1.1). My point of view is to suppose that this action is the correct action to describe gravity in a spacetime of any dimension, signature, and topology, without the need to augment it with any additional boundary terms put in by hand. It is our job now to simply analyze it and extract the physics it contains.

This action is not really new. In the $d = 4$ Lorentzian case a bit of index manipulation shows that $I^{(1)}[e]$ is the same action proposed by Goldberg [23], which he uses to derive a formalism equivalent to the Ashtekar-variables approach [24]. However, Goldberg's derivation of $I^{(1)}[e]$ is different, and is based on the so-called Sparling-Thirring forms. These forms are interesting in themselves because they satisfy a relationship that expresses the Einstein tensor in terms of an energy-momentum pseudotensor and a superpotential [23]. Related to this fact is work by Lau [25,26] who uses this Goldberg action to derive an Ashtekar-variables reformulation of the metric theory of quasilocal stress-energy-momentum originally due to Brown and York [27]. So it would seem that $I^{(1)}[e]$ is particularly Ashtekar-variables-friendly, but I will not explore this aspect of the formalism here. Finally, $I^{(1)}[e]$ is similar to the first order action originally proposed by Einstein [28], but differs from it by a tetrad expression that is not expressible in terms of the metric itself. The main contribution of this paper is the recognition that $I_{\partial M}[e]$ contains a boundary action describing new physical gravitational degrees of freedom living on ∂M, and the derivation of what I think are interesting and significant consequences of this fact.

[2]Taken in context, there should be no confusion between this g and the spacetime metric, g.

Regarding symmetries, we first observe that the proposed action is obviously invariant under diffeomorphisms which preserve the boundary, and no motivation will present itself to contemplate diffeomorphisms outside of this restriction. Secondly, the proposed action is invariant under local frame rotations except at the boundary, where the undifferentiated W in $I_{\partial M}[e]$ breaks this symmetry. A broken symmetry such as this might at first be thought of as a defect of the action, but it is precisely this feature which allows us to apply Carlip's program [7, 13, 14] to arrive at gravitational boundary degrees of freedom, which will eventually lead to the microstates I am suggesting might be responsible for black hole entropy. And this is the principal reason for going to an orthonormal frame (rather than metric) formulation of gravity.

So, following in the spirit of Carlip's work, let us parametrize the frame as $e^a = U^a{}_b \hat{e}^b$, where $\hat{e}^b$ is a gauge-fixed frame (with corresponding gauge-fixed connection, $\hat{\omega}^a{}_b$), and all frame rotation degrees of freedom are included in $U = \text{matrix}\,(U^a{}_b) \in G$. In terms of this matrix notation the parametrization reads

$$E = U\hat{E}U^{-1}\,, \quad W = U\hat{W}U^{-1} + UdU^{-1}\,, \tag{1.5}$$

and the action accordingly splits:

$$I^{(1)}[e] = I^{(1)}[\hat{e}] + I_B[U;\hat{e}]\,, \tag{1.6}$$

into a gauge-fixed action, $I^{(1)}[\hat{e}]$, plus (what I shall interpret as) a boundary action:

$$I_B[U;\hat{e}] = \frac{1}{2\kappa}\int_{\partial M} \mathrm{Tr}\,(U^{-1}dU \wedge \hat{E})\,. \tag{1.7}$$

I will analyze these two objects in turn.

2 The Gauge-Fixed Action, $I^{(1)}[\hat{e}]$

The gauge-fixed action depends on the choice we make for the gauge-fixed frame on ∂M. But of course all choices are equivalent, at least at the classical level, since a different choice can be absorbed into a change of variables of the U degrees of freedom. So the choice is merely a matter of convenience, rather than 'putting in physics by hand.' It turns out that a convenient choice is to gauge-fix one leg of the vector frame, denoted as $\hat{e}_\perp$, to equal the unit normal, n, everywhere on ∂M.[3] A short calculation reveals that with this choice the gauge-fixed action is

$$I^{(1)}[\hat{e}] = \frac{1}{2\kappa}\int_M \epsilon\, R + \pi_n \frac{1}{\kappa}\int_{\partial M} \epsilon_{\partial M}\, K\,, \tag{2.1}$$

where K is the trace of the extrinsic curvature of the boundary, $\epsilon_{\partial M} = i_n \epsilon$ is the boundary volume form, and $\pi_n = n \cdot n = \pm 1$, the sign depending on the relative signatures of the metrics on M and ∂M. This result is independent of how we rotate the remaining legs of

[3] $I^{(1)}[\hat{e}]$ contains some interesting physics when the boundary is null, but since I have not yet analyzed the boundary action for this case, which in my principal focus here, we will restrict ourselves to the case where ∂M is nowhere null.

the gauge-fixed vector frame ($\hat{e}_a$, $a \neq \perp$) in the tangent space of ∂M. We see that $I^{(1)}[\hat{e}]$ is just the standard Einstein-Hilbert action appropriate for holding fixed, in the action principle, the boundary $(d-1)$-geometry [28]. A similar result for the Goldberg action was first shown by Lau [26].

But this is not the end of the story. For example, suppose we consider a Lorentzian spacetime in which a spacelike section of ∂M (with unit normal u) joins a timelike section (with unit normal n) in a $(d-2)$-dimensional 'corner,' $\mathcal{C}$. If we gauge-fix an appropriate leg of the vector frame on each section as described in the previous paragraph (say $\hat{e}_0 = u$ and $\hat{e}_1 = n$ on the respective sections) we pick up a trace K term from each, as in (2.1), but since on $\mathcal{C}$ $n \cdot u \neq 0$, in general, the frame is double-valued on $\mathcal{C}$. It is easy to render it single-valued simply by introducing a finite frame rotation (in this case a boost) in an infinitesimal neighborhood of $\mathcal{C}$, which has the effect of augmenting the right hand side of (2.1) with an additional 'corner term' of the form

$$\frac{1}{\kappa}\int_{\mathcal{C}} \epsilon_{\mathcal{C}} \operatorname{arcsinh}(n \cdot u)\,, \tag{2.2}$$

where $\epsilon_{\mathcal{C}}$ is the corner volume form. Again, in the context of the Goldberg action, this sort of calculation was first done by Lau [26] so I will not reproduce a similar calculation here. It is easy to demonstrate that, when ∂M has corners, precisely such corner terms are required to make the Einstein-Hilbert action well defined. This has been known for some time [29,30] (see also Ref. [31]), and it is satisfying that these corner terms arise *naturally* in $I^{(1)}[\hat{e}]$ simply by demanding single-valuedness of the frame. Indeed, this is one reason I am suggesting that $I^{(1)}[e]$ might be 'universal'—recall my comment immediately following (1.4). Furthermore, this nice property of $I^{(1)}[\hat{e}]$ provides motivation to take seriously the other piece of $I^{(1)}[e]$, namely $I_B[U;\hat{e}]$ in (1.6), which will be the principal focus of this paper.

Now let us turn our attention to Euclidean black hole spacetimes, assuming a topology $M = R^2 \times S^{d-2}$ with boundary $\partial M = S^1 \times S^{d-2}$ at infinity. In this case ∂M has no corners and there appears to be no problem in gauge-fixing, say, $\hat{e}_0 = n$ (and $\hat{e}_1 = t$, a Euclidean time flow unit vector tangent to ∂M), to obtain (2.1). However, now a subtlety of a different sort arises: This choice of gauge is analogous to using the frame $\hat{e}_0 = \partial/\partial r$, $\hat{e}_1 = (1/r)\partial/\partial\phi$ on a flat disk with polar coordinates (r, ϕ), the disk being analogous to the R^2 sector of M. On the boundary $(r = 1)$ this is all right, but the frame is multivalued at the origin, and no other choice of frame in the interior (smoothly connected to our choice on the boundary) can avoid such a point of multivaluedness. Now, although this disk example falls short of a proof, it seems highly plausible that a similar problem obtains in the black hole case, where this type of multivaluedness must exist on some set of points in the interior of M, say the bifurcation sphere $\{0\} \times S^{d-2}$. (And even if one could prove that this is not *necessarily* the case, such a choice of frame is certainly possible, and arguably natural.) One might wonder, "So what? Such frames are used all the time, and their multivaluedness is no more problematic than a mere coordinate singularity." I argue that this is not so: First, we learned in the Lorentzian case that demanding single-valuedness of the frame is necessary to obtain the correct corner terms. And secondly, if the proposed action is to be taken seriously as a description of gravity based not on the metric, but on observers' frames, it seems physically unreasonable to allow more than one frame attached to a given spacetime point.

I think these arguments provide sound motivation for excising the bifurcation sphere from the spacetime, which obviously removes the multivaluedness problem, and is the simplest (but not the only)[4] solution. This subtlety is important because now the spacetime becomes an annulus cross S^{d-2}, with both an outer and an inner $S^1 \times S^{d-2}$ boundary, the latter bounding a 'thickened bifurcation sphere.' This means there will be boundary degrees of freedom (discussed in the next section) on both outer and inner boundaries, which on quantization will give rise to two independent sets of microstates. If Γ_o and Γ_i denote respectively the degeneracies of these microstates, then the total degeneracy of microstates will be the product, $\Gamma_o\Gamma_i$. And according to statistical mechanics, the total entropy will be $\ln\Gamma_o + \ln\Gamma_i$ (with Boltzmann's constant set to one). If the boundary theory I am suggesting here works at all we would expect $\ln\Gamma$ to be proportional to the volume of a $(d-2)$-sphere cross section of the boundary, which means that $\ln\Gamma_o$ will be divergent. However, it is well known that the action in (2.1) is also divergent when evaluated on a Euclidean black hole spacetime (due to the trace K term integrated over the outer boundary at infinity), and that the *physical* action is obtained by subtracting the same action evaluated on a suitable vacuum spacetime, yielding a finite result [27, 22, 32, 33, 34]. The analogous regularization procedure here involves subtracting the entropy of the vacuum spacetime. This will consist of the *same* $\ln\Gamma_o$ term (since, by definition, a "suitable" vacuum spacetime has the same intrinsic outer boundary geometry as the black hole), but no $\ln\Gamma_i$ term. (We may still need to excise some set of points from the interior of the vacuum spacetime to obtain a single-valued frame, but a little thought shows that the boundary theory resulting from this excision is trivial—I will elaborate on this in Section 6.) So the physical entropy is just $\ln\Gamma_i$, precisely what we want, or might expect.

This result may be viewed as a direct response to a speculation by Carlip and Teitelboim (the last sentence in Ref. [35]) in which they suggest that the numerical factor relating the entropy of a Euclidean black hole to the area of its bifurcation sphere might have its origin in microstates living on the boundary of a thickened bifurcation sphere. We will see later (equation (6.3)) that this numerical factor is deeply connected with the nature of the quantized boundary theory.

Furthermore, consider the following. It is well known that the topology of a manifold is intimately connected with the index of certain vector (or one-form) fields with isolated singularities in the manifold and on its boundary [36, 37]. Insofar as the vector (or one-form) frame field (rather than the metric) is the fundamental object out of which $I^{(1)}[e]$ is constructed, and it is precisely a certain type of isolated singularity (the multivaluedness) of

[4] Returning to the disk example, it is also possible to begin with the choice $\hat{e}_0 = \partial/\partial x$, $\hat{e}_1 = \partial/\partial y$ in Cartesian (x,y) coordinates, so that at the point (1,0) on the boundary $\hat{e}_0 = n$. Then as one circles counterclockwise around the boundary the frame is rotated such that $\hat{e}_0$ remains in the normal direction. Through the last infinitesimal segment to complete the circuit a clockwise 2π rotation is executed to return the frame to its original orientation. It is clear that this operation on the boundary can be continued smoothly into the interior, and the resulting frame is single-valued everywhere. The 2π rotation in an infinitesimal neighborhood of the boundary point (1,0) produces a 'topological' (as opposed to corner) term, which in the black hole case turns out to equal $-2\pi/\kappa$ times the volume of the $(d-2)$-sphere fiber at the point (1,0). This seems unsatisfactory in that our starting point, (1,0), becomes a preferred point (unless the spacetime admits a Euclidean time Killing vector field on ∂M). For this, and other reasons, we will adhere to the solution of the multivaluedness problem presented in the main text (exception: see footnote 5).

this frame field which motivated excising the bifurcation sphere, leading to the above $\ln \Gamma_i$ contribution, it is tempting to speculate that this contribution is really of topological origin. What I mean is that this appears to be an example of a general mechanism whereby $I^{(1)}[e]$ encodes physical consequences of the topology of the spacetime. Stated even more strongly, the entropy of a black hole is a topological phenomenon. This point of view is advocated by Gibbons and Hawking [4] (see also Refs. [5, 33, 38, 39]), but notice that what is suggested here is something more: it is not merely the Bekenstein-Hawking entropy (one-quarter the horizon area) we are discussing, but rather the quantum microstates responsible for that entropy.

The 'excision principle' introduced above is crucial to the physics we are considering. I have given a physical justification for it, now let me provide a more mathematical one. To be concrete let us consider the example of the Euclidean-Schwarzschild spacetime (for simplicity without a conical singularity), which is topologically $R^2 \times S^2$. It is easy to verify for this metric that equation (1.2) is satisfied at every point in M. Now when we integrate this equation over M the left hand side is of course zero, and the right hand side gives a volume term (call it V) plus a surface term (call it S), the latter an integration over ∂M (proportional to $I_{\partial M}[e]$). We require $V + S = 0$, but the question arises, "What is ∂M?" If we assume that the Euclidean-Schwarzschild spacetime has a boundary only at infinity we get a wrong result: $V + S \neq 0$. Perhaps somewhat surprisingly, it turns out that to get the correct result we must assume that ∂M consists of the usual boundary at infinity *plus* the boundary (call it ∂M_*) of the following suitably 'thickened' set of points: all points comprising the bifurcation sphere, and the north and south poles of the two-sphere at each point of R^2. What is happening here? Having chosen one leg of the tetrad in the radial direction and another in the Euclidean time direction, this pair of legs is multivalued on the bifurcation sphere, as discussed previously. Furthermore, the two-sphere is not parallelizable: if the remaining two legs of the tetrad are chosen in the $\partial/\partial\theta$ and $\partial/\partial\phi$ directions, respectively (where θ and ϕ denote standard spherical coordinates), then this pair of legs is multivalued at the two poles of the sphere. The lesson from this example is that the action $I^{(1)}[e]$ *tells us* what ∂M must be: in order for the $\mathrm{Tr}\,(W \wedge W \wedge E)$ volume integral in (1.4) to equal $I^{(2)}[g(e)] + I_{\partial M}[e]$ we are *forced* to excise from M all points at which the frame is multivalued. This set of points, being of measure zero, does not affect the volume integral, but does affect the boundary term $I_{\partial M}[e]$. As a consequence, microstates live on all points of ∂M_* (as well as on the boundary at infinity, but as argued above, regularization renders these non-physical). In the Euclidean-Schwarzschild example it is easy to show that the boundary theory on the portion of ∂M_* corresponding to excision of the poles of S^2 is trivial, but that on the portion of ∂M_* which is the boundary of the thickened bifurcation sphere is not, and it is here that the physically relevant microstates in this example live. A similar phenomenon occurs in each of several examples I have studied, and thus it appears that it may be generic.

Furthermore, the usual value of the regularized on-shell Euclidean action (taking into account only the boundary at infinity) is changed when we take into account ∂M_*, and consequently so is the thermodynamical entropy calculated from the zero-order partition function. (Generically this is so, but not, it turns out, in the Euclidean-Schwarzschild case, due to some 'miraculous' cancellations.) It has been emphasized by Brill and Hayward that

the action for a spacetime is not necessarily invariant under topological identifications of isometric surfaces on its boundary; a finite action is associated with certain identification surfaces [31]. Their analysis involves precisely the type of corner terms discussed earlier, and it seems likely that their findings are closely connected to what is happening here with $I^{(1)}[e]$.

In summary, let me reemphasize that necessary singularities (multivaluedness) of the vector or one-form frame are associated with a nontrivial topology of M, and $I^{(1)}[e]$ has a remarkable built-in sensitivity to this measure of topology, which manifests itself in an excision principle, and which in turn dictates the location where the physically relevant microstates live.[5] Thus, one might say that entropy is a topological phenomenon not only at the level of thermodynamics, but also at the level of statistical mechanics.

3 The Action Principle

For clarity of exposition in this section we shall restrict ourselves to the 'regular' sections of ∂M, ignoring corners if any, and, where necessary, we assume that we have excised a suitable set of points from M such that, as discussed in the previous section, we can gauge-fix $\hat{e}_\perp = n$ everywhere on ∂M. The $(d-1)$ vectors $\hat{e}_i$ (i running over the values of a, $a \neq \perp$) are everywhere tangent to ∂M. That the vector frame $\hat{e}_a$ is gauge-fixed means that, when computing variations, any occurence of the combination $\hat{e}_{[a} \cdot \delta \hat{e}_{b]}$ is set to zero, as it represents a rotation degree of freedom already accounted for in $U^a{}_b$. Finally, let $\mu, \nu, \ldots$ denote spacetime tensor indices referred to a set of local coordinates in M (or their restriction to ∂M), which are raised and lowered with the spacetime metric $g_{\mu\nu}$.

Varying the action $I^{(1)}[e]$ in (1.6) with respect to $\hat{e}$ and U we obtain

$$\delta I^{(1)}[e] = \frac{1}{2\kappa}\int_M \epsilon\,(R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu})\,\delta g^{\mu\nu}(\hat{e}) - \frac{1}{2\kappa}\int_{\partial M} \mathrm{Tr}\,(dE\,\delta U U^{-1}) + \pi_n \frac{1}{2\kappa}\int_{\partial M} \epsilon_{\partial M}\,(\Pi_{\mu\nu} + \kappa T_{\mu\nu})\,\delta h^{\mu\nu}(\hat{e})\,. \tag{3.1}$$

So for the action to be extremized we require, first, that the metric $g_{\mu\nu}(\hat{e})$, constructed out of $\hat{e}^a$, satisfy the vacuum Einstein equations at every point in M.[6] The second integral comes from varying the boundary action, $I_B[U;\hat{e}]$, with respect to U, and it will vanish if and only if the boundary degrees of freedom, U, satisfy the Euler-Lagrange equations

$$0 = dE \downarrow_{\partial M} = d\,(U\hat{E}U^{-1}) \downarrow_{\partial M}\,, \tag{3.2}$$

[5]Recently Strominger [40] has given a new microscopic derivation of the BTZ black hole entropy based on the observation [41] that the asymptotic symmetry group of anti-de Sitter space is infinite dimensional, generated by (two copies of) the Virasoro algebra. His derivation suggests that the microscopic degrees of freedom might reside at infinity, rather than at the horizon. (Or perhaps in this special case both possibilities can be shown to be equivalent.) This may be a good example in which to test the alternative to the excision principle mentioned in footnote 4, but I will not do so here.

[6]Notice that excising a set of points from M naturally allows for the possibility of introducing conical singularities at these points without affecting the bulk term in (3.1), but I will not discuss this generalization here.

where the symbol $\downarrow_{\partial M}$ denotes pullback of forms to ∂M. Recall that $\hat{E}$ is a g-valued $(d-2)$-form (constructed out of the gauge-fixed frame); $E = U\hat{E}U^{-1}$ is its orbit under the action of $U \in G$. These boundary equations for U are deceptively simple-looking—in fact they are highly nontrivial. I will not present any results of the analysis of them in this paper, except for some brief comments in Section 4.

In the third integral on the right hand side of (3.1), $h_{\mu\nu}(\hat{e})$ is the induced boundary metric, constructed out of $\hat{e}^i$. $\Pi_{\mu\nu} = K_{\mu\nu} - Kh_{\mu\nu}$ is the usual gravitational momentum canonically conjugate to $h^{\mu\nu}$ (and $K_{\mu\nu} = h_\mu{}^\rho \nabla_\rho n_\nu$ is the extrinsic curvature of the boundary). It is a fact that the boundary action $I_B[U;\hat{e}]$ depends on $\hat{e}$ only through $h_{\mu\nu}(\hat{e})$. So varying $I_B[U;\hat{e}]$ with respect to $\hat{e}$ amounts to calculating the energy-momentum tensor for the U degrees of freedom, the result being:

$$T^{\mu\nu} = \frac{1}{\kappa}[\eta^{ij}\eta^{kl} - \eta^{ik}\eta^{jl}]\, i_{\hat{e}_i}(U^{-1}dU)_{j\perp}\, \hat{e}_k{}^{(\mu}\hat{e}_l{}^{\nu)}\,, \tag{3.3}$$

which has the standard '$p\partial q$' form. (Since the boundary action is linear in first derivatives of U, U is already a phase space variable.) As remarked earlier, the action $I^{(1)}[e]$ is invariant under diffeomorphisms which preserve ∂M. This means that on-shell we must have $D_\nu(\Pi^{\mu\nu} + \kappa T^{\mu\nu}) = 0$, where D_ν is the covariant derivative operator induced on ∂M. But we already know that $D_\nu \Pi^{\mu\nu} = 0$ on-shell—this is just the momentum constraint of general relativity on the $(d-1)$-surface ∂M. So we must have $D_\nu T^{\mu\nu} = 0$ on-shell. Indeed, this can be explicitly shown to follow from the boundary equations (3.2) and the above definition of $T^{\mu\nu}$. So when $h_{\mu\nu}(\hat{e})$ admits Killing vectors one can construct corresponding boundary conserved charges, such as the total energy, momentum, or angular momentum corresponding to a given solution U.

Now there are two ways to interpret the third integral on the right hand side of (3.1) with regard to extremizing the action. The first is to suppose, as in the standard Einstein-Hilbert action, that one holds fixed the boundary $(d-1)$-geometry, i.e., $h_{\mu\nu}(\hat{e})$ up to diffeomorphisms. (In this case the 'total momentum' $\Pi_{\mu\nu} + \kappa T_{\mu\nu}$ is not restricted on ∂M.) Following in the spirit of Carlip's work [7, 13, 14] the following statistical mechanical interpretation of the boundary theory can then be given. One solves the vacuum Einstein equations for $g_{\mu\nu}(\hat{e})$ in M to obtain a 'macrostate,' for instance a black hole solution. Information about this macrostate, for example the black hole mass and angular momentum, enters the boundary action $I_B[U;\hat{e}]$ through $\hat{e}$ (or equivalently, the boundary metric $h_{\mu\nu}(\hat{e})$). $h_{\mu\nu}(\hat{e})$ is thought of as a fixed background metric on ∂M, whereas the boundary degrees of freedom, U, are dynamical, and, on quantization, lead to the microstates responsible for the entropy of this macrostate. As for a physical interpretation of the boundary degrees of freedom, I think the following one is both simple and compelling: If there is *any* sense in which a boundary in spacetime is physically meaningful[7] (event horizon, apparent horizon, thickened bifurcation sphere [35], membrane model of horizon [42], black hole complementarity approach [43], 't Hooft's "brick wall" approach [44] (and later refinements [45]), black hole in a box, e.g. [22], quasilocal quantities associated with spatially bounded regions, e.g. [27], boundary at infinity, and so on), then the orientation of an observer's frame relative to this boundary (encoded in U) is *likewise* physically meaningful.

[7]See section III of Ref. [14] for further discussion on this point.

A second way to extremize the action in (3.1) is what might be called a 'no boundary condition proposal.' One allows $h_{\mu\nu}(\hat{e})$ to vary freely on the boundary (hence no boundary conditions), in which case one is forced to make the identification

$$T^{\mu\nu} = -\frac{1}{\kappa}\Pi^{\mu\nu}\,. \tag{3.4}$$

This identification seems natural in light of the work by Brown and York on quasilocal quantities associated with spatially bounded regions in general relativity, wherein they define essentially the right hand side of (3.4) as the boundary gravitational energy-momentum tensor [27]. So the identification (3.4) in some way reflects a coupling between bulk ($g_{\mu\nu}(\hat{e})$) and boundary (U) degrees of freedom. (In a different context, a coupling of this sort is discussed in Ref. [8].)

I have already mentioned that $D_\nu T^{\mu\nu} = 0$ follows from the boundary equations (3.2), and so under the identification (3.4), the boundary equations imply the momentum constraints of general relativity. (Which remains true if we replace ∂M in our analysis with any $(d-1)$-dimensional now initial-value surface in M.) Applied in the simplest case, the ($d = 3$) Euclidean BTZ black hole [6] (for which, of course, we augment $I^{(1)}[e]$ with a cosmological term), it turns out that the U boundary equations also imply the Hamiltonian constraint of general relativity, and there are indications that this may be true in general. Even more surprising, these same indications hint that the converse may also be true: under the identification (3.4) our boundary theory for U may be *equivalent* to the initial-value constraints of general relativity. To appreciate the potential significance of this, bear in mind that essentially the full content of general relativity is encoded in its initial-value constraints, in that, given a good set of initial data on a Cauchy slice, the full spacetime is generated canonically by these same constraints. And to the extent that our boundary theory for U can be quantized, and in principle, at least, solved (see Section 6), proving the above-suggested equivalence may pave the way to one solution of quantum gravity. But insofar as this 'no boundary condition proposal' is still somewhat speculative (work in progress), in the remainder of this paper I will adhere to the interpretation of the action given in the previous paragraph, namely that $h_{\mu\nu}(\hat{e})$ is a fixed background metric, and only U is dynamical on ∂M.

4 The Boundary Theory Phase Space

In the previous sections I have attempted to carefully describe my motivations for studying the action $I^{(1)}[e]$, and in the remainder of the paper I will quickly outline some of the main results concerning its most interesting aspect, namely the boundary theory it contains.

Up to an unimportant sign, which depends on the signatures of the spacetime and its boundary, the symplectic structure of the boundary theory has a coadjoint orbit form given by

$$\omega = \pm\frac{1}{2\kappa}\int_{\partial\Sigma} \epsilon_{\partial\Sigma}\,\mathrm{Tr}\,\{\hat{T}(U^{-1}\delta U)(U^{-1}\delta U)\}\,. \tag{4.1}$$

Here $\partial\Sigma$ is a $(d-2)$-dimensional Cauchy surface in ∂M, which can be thought of as the intersection of Σ, a $(d-1)$-dimensional Cauchy surface in M, with the boundary ∂M. Using

(3.2) it is easy to show that ω is invariant under deformations of $\partial\Sigma$ connected to the identity, which is to say that this is a covariant description of the boundary theory phase space. We will focus here on the Euclidean black hole (with bifurcation $(d-2)$-sphere excised—see Section 2), which is topologically an annulus cross S^{d-2}. Choosing a constant (Euclidean) time surface $\Sigma = [r_i, r_o] \times S^{d-2}$, we have $\partial\Sigma = S_i^{d-2} \cup S_o^{d-2}$, where S_o^{d-2} is a large sphere at infinity and S_i^{d-2} is a sphere cross section of the boundary of the thickened bifurcation sphere. At any instant of time, U is a map from $\partial\Sigma$ into the rotation group, G. Under time evolution this map changes, and its history is the G-valued function U on ∂M. For example, in the $d=3$ case, U is (two copies of) a time dependent map from S^1 (S_i^1 and S_o^1) into $SO(3)$, also known as the loop group of $SO(3)$. $\epsilon_{\partial\Sigma}$ is the volume form induced on $\partial\Sigma$, and depends on 'macrostate' information encoded in $h_{\mu\nu}(\hat{e})$. And finally, $\hat{T}$ is a fixed element of the Lie algebra, g, which is (or rather $\mp\epsilon_{\partial\Sigma}\hat{T}$ is) the pullback of $\hat{E}$ to $\partial\Sigma$. The heart of the boundary theory is the orbit, $U\hat{T}U^{-1}$, of $\hat{T}$ under the action of U.

Now let H denote the isotropy subgroup of $\hat{T}$, i.e., the elements V of G satisfying $V\hat{T}V^{-1} = \hat{T}$. Introducing the coset decomposition

$$U = \tilde{U}V\,; \quad U \in G\,,\ V \in H\,,\ \tilde{U} \in G/H \tag{4.2}$$

the symplectic structure in (4.1) reduces to

$$\omega = \pm\frac{1}{2\kappa}\int_{\partial\Sigma} \epsilon_{\partial\Sigma}\, \mathrm{Tr}\,\{\hat{T}(\tilde{U}^{-1}\delta\tilde{U})(\tilde{U}^{-1}\delta\tilde{U})\}\,, \tag{4.3}$$

which means that the reduced (or physical) phase space consists of the set of all maps $\tilde{U}$ from $\partial\Sigma$ into the coset space G/H.[8] The image of $\partial\Sigma$ in G/H determined by a given $\tilde{U}$ can be thought of as the classical state of the system at a given instant of time, and this image evolves with time. For the Euclidean black hole the maps we are dealing with are those from S^1 into $SO(3)/SO(2) \cong S^2$ ($d=3$ case), and S^2 into $SO(4)/(SO(2)\times SO(2)) \cong S^2 \times S^2$ ($d=4$ case). In general, $G/H = SO(d)/(SO(2)\times SO(d-2))$, an oriented Grassmann manifold, with corresponding changes from compact to noncompact groups when the spacetime or boundary have non-Euclidean signature. The dimension of G/H is $2(d-2)$, twice the dimension of $\partial\Sigma$, and our phase space (which is of course infinite-dimensional) can be thought of as even-dimensional for all d.

What does this symmetry H correspond to physically? Let the gauge-fixed vector frame $\hat{e}_a$ consist of: $\hat{e}_\perp$, normal to ∂M; $\hat{e}_\top$, tangent to ∂M and normal to $\partial\Sigma$—a time flow unit vector; and the remaining $(d-2)$ unit vectors, $\hat{e}_\alpha$, tangent to $\partial\Sigma$. Then the $SO(2)$ component of H corresponds to rotations involving just $\hat{e}_\perp$ and $\hat{e}_\top$; the $SO(d-2)$ component to rotations involving just the $\hat{e}_\alpha$. This means the boundary theory considers as physical only an observer's orientation *relative to* $\partial\Sigma$, for instance the bifurcation sphere, or the large sphere at infinity.

[8]When U is a rotation which preserves the normal, i.e., $e_\perp = \hat{e}_\perp$, the boundary action is independent of the remaining free rotation angle(s). This symmetry reduces the physical phase space further, but only by the removal of a set of points of measure zero. Since our goal is to evaluate the degeneracy of microstates, which at the semiclassical level corresponds to calculating a volume in the physical phase space, this subtlety should not significantly affect our results, and I will ignore it here.

The maps from $\partial\Sigma$ into G/H have a semi-direct product type of structure consisting of reparametrizations of the image and deformations of the image normal to itself (normal with respect to the natural metric on the Grassmann manifold). Applied in the simplest case, the ($d = 3$) Euclidean BTZ black hole [6], it turns out that the components of the energy-momentum tensor $T^{\mu\nu}$ in (3.3) canonically generate both reparametrizations and deformations, in certain combinations. Their Poisson algebra has a Virasoro algebra piece (presumably corresponding to the reparametrizations), and an additional rather complicated piece depending on the acceleration, or extrinsic curvature of the image—in this ($d = 3$) case a closed curve S^1 in S^2 (presumably corresponding to the deformations). The Hamiltonian, equal to the invariant length of the image curve, canonically generates reparametrizations (only), the rate of reparametrization depending on the extrinsic curvature of the image curve at the point in question. The simplest solutions to (3.2) are curves of constant extrinsic curvature (e.g., latitudes of the sphere), this constant being restricted to an infinite but discrete set of values (discrete because the solutions must be periodic in Euclidean time, with period equal to the inverse temperature of the black hole). These solutions have the form of free fields propagating on ∂M. The case of non-constant extrinsic curvature, which can be solved only implicitly, describes an apparently much larger sector of the solution space in which the solutions generically exhibit a "shock discontinuity" phenomenon of the type discussed in Ref. [46] (and familiar to those studying fluid dynamics).

Returning to the general case we make the following interesting observation: The U boundary degrees of freedom, which I am suggesting are ultimately responsible for black hole entropy, are intimately associated with the group *Diff*$(\partial\Sigma)$, in particular the diffeomorphism group of the bifurcation $(d-2)$-sphere. This type of result was speculated by Carlip in the conclusion section of Ref. [14], and it is satisfying to see a concrete realization of this idea. And notice that the diffeomorphism symmetry of the action $I^{(1)}[e]$ is not broken in order to achieve this. Nor do diffeomorphisms of the boundary play any role as gravitational boundary degrees of freedom. (I mention this because there are interesting discussions to the contrary which suggest that diffeomorphisms of the boundary, perhaps even those which do not preserve the boundary, might represent gravitational boundary degrees of freedom [8,14,47]. While this might be true, it is not obvious to me how the results here could be connected with that idea, except perhaps through the 'no boundary condition proposal' discussed in Section 3.) Anticipating some possible confusion on this point, let me restate that the action is invariant under diffeomorphisms of e (read: *simultaneous* diffeomorphisms of U and $\hat{e}$) which preserve ∂M. On the other hand, while it is true that a reparametrization of the image of $\partial\Sigma$ in G/H is related to a diffeomorphism of the scalar field U on ∂M, since $\hat{e}$ is not simultaneously undergoing the same diffeomorphism transformation the action is in general not invariant. In other words, reparametrizations of the image are not, in general, (diffeomorphism) symmetries of the action, but in fact are physical degrees of freedom.

5 Generalized Kač-Moody Algebras and Virasoro Operators

$\partial\Sigma$ may consist of a number of disconnected components; in an abuse of notation I will use $\partial\Sigma$ to denote any one of these components, which are typically $(d-2)$-spheres. We introduce

a mode basis, f_M, M a collective index of integers, which satisfy the following relations:

$$\int_{\partial\Sigma} \epsilon_{\partial\Sigma}\, \bar{f}_M f_N = \delta_{MN}\, \mathrm{Vol}\,(\partial\Sigma)\,, \quad \text{where } \mathrm{Vol}\,(\partial\Sigma) = \int_{\partial\Sigma} \epsilon_{\partial\Sigma}\,, \tag{5.1}$$

$$f_M f_N = \sum_P \bar{C}_{MNP}\, f_P\,. \tag{5.2}$$

Here $\bar{f}_M$ denotes the complex conjugate of f_M in case the mode basis is complex. The C_{MNP} are mode structure constants analogous to the familiar Clebsch-Gordon coefficients used in the quantum mechanical treatment of angular momentum, which are calculated using (5.1).[9] Recall that the heart of the boundary theory is the orbit of $\hat{T}$ under the action of U: there is a Lie algebra-valued current, J, proportional to $U\hat{T}U^{-1}$ $(= \tilde{U}\hat{T}\tilde{U}^{-1})$. Defining the modes of this current as

$$J_M = \mp\frac{1}{\kappa}\int_{\partial\Sigma} \epsilon_{\partial\Sigma}\, \bar{f}_M\, (\tilde{U}\hat{T}\tilde{U}^{-1})\,, \tag{5.3}$$

it can be shown that the symplectic structure given in (4.3) is equivalent to the Poisson bracket algebra

$$\{J_M^A, J_N^B\} = -f^{AB}{}_C \sum_P C_{MNP}\, J_P^C\,, \tag{5.4}$$

with the currents being subject to the quadratic constraints

$$L_P = \left(\frac{\mathrm{Vol}\,(\partial\Sigma)}{\kappa}\right)^2 \delta_{PP_0}\,, \quad \text{with } L_P = \frac{1}{\hat{N}}\sum_{M,N} C_{PMN}\, N_{AB}\, \bar{J}_M^A J_N^B\,. \tag{5.5}$$

My Lie algebra basis notational conventions are $[T_A, T_B] = f_{AB}{}^C T_C$ and $\mathrm{Tr}\,(T_A T_B) = -2N_{AB}$, where N_{AB} and its inverse are used to raise and lower the Lie algebra indices. These indices run from 1 to $d(d-1)/2$. $\hat{N}$ is defined to be $-\mathrm{Tr}\,(\hat{T}\hat{T})/2$. P_0 is the unique value of the collective index P such that $f_{P_0} = 1$, the constant mode. With these conventions we have $C_{P_0 MN} = \delta_{MN}$, so that L_{P_0} is real and positive definite (at least in the Euclidean case).

The constraints in (5.5) follow from the fact that $\mathrm{Tr}\,(JJ)$ equals a constant: the J_M^A comprise an overcomplete set of phase space coordinates. As a simple analogy, consider a two-sphere of radius R (with spherical coordinates (θ, ϕ)) as a phase space with canonical coordinates $p = -R\cos\theta$, $q = \phi$, such that $\{p, q\} = 1$. It is often more convenient to use instead the overcomplete set of phase space coordinates $J^1 = R\sin\theta\cos\phi$, $J^2 = R\sin\theta\sin\phi$, $J^3 = R\cos\theta$ satisfying an angular momentum type of algebra $[J^1, J^2] = J^3$, etc. (the analogue of (5.4)), and subject to the constraint $|J|^2 = R^2$ (the analogue of (5.5)). Notice that, like its analogue R, $\mathrm{Vol}\,(\partial\Sigma)/\kappa$ in (5.5) sets the *scale* of the phase space.

These constraints are similar to those encountered in string theory, where the L_{P_0} constraint here is analogous to what is known there as the mass-shell condition [48]. In the case $d = 4$ let M_s be the string mass and let α' denote what is known as the universal Regge

[9] Notice that the C_{MNP} might encode information about the 'macrostate' not already contained in $\mathrm{Vol}\,(\partial\Sigma)$, such as information about a conformal 'weight factor' in the measure, or the Teichmüller parameters of the metric on $\partial\Sigma$, and that this information will be reflected in the quantization. I have not yet investigated this possibility in any detail.

slope parameter (inversely proportional to the string tension). Then the role of $\sqrt{\alpha'}M_s$ in string theory is played here by $\mathrm{Vol}\,(\partial\Sigma)/\kappa$; the latter can be taken to be proportional to the area of the bifurcation sphere in Planck units, i.e., GM_{bh}^2, where G is Newton's constant and M_{bh} is the black hole mass. As argued in Ref. [49] the string coupling should be chosen such that $\sqrt{\alpha'}/G \sim M_{bh}$, which means $M_s \sim M_{bh}$. This identification of masses suggests that the black hole can be viewed as an excited string state [20]. Now, the mass-shell condition determines M_s in terms of the internal vibrational modes of the string. Based on the preceding discussion it thus seems reasonable to suggest that the L_{P_0} constraint, in some similar way, associates the mass of the black hole (or the area of its bifurcation sphere) with the U vibrational modes on its horizon (more precisely, the boundary of its thickened bifurcation sphere). Additional hints that the analysis here might have hidden connections with string theory are discussed in Section 6.

Let us see what (5.4) and (5.5) look like when $d = 3$. In this case take $\partial\Sigma = S^1$; then $f_M = f_m(\phi) = e^{im\phi}$, m an integer, $C_{mnp} = \delta_{m+n,p}$, and (5.4) reduces to the (classical) Kač-Moody algebra

$$\{J^A_m, J^B_n\} = -f^{AB}{}_C J^C_{m+n}\,, \tag{5.6}$$

and the constraints in (5.5) take the form

$$L_0 = \left(\frac{\mathrm{Vol}\,(\partial\Sigma)}{\kappa}\right)^2\,; \quad L_m = 0\,,\ m > 0\,; \quad L_m = \frac{1}{\hat{N}}\sum_n N_{AB}\,\bar{J}^A_{n-m}J^B_n\,. \tag{5.7}$$

(The constraints $L_m = 0$, $m < 0$ are accounted for by the reality condition $\bar{L}_m = L_{-m}$, which follows from $\bar{f}_m = f_{-m}$. Just as in the Gupta-Bleuler treatment of electrodynamics, this observation is necessary to make sense of such constraints at the quantum level—see, e.g., Section 2.2 of Ref. [48].) The L_m are known as Virasoro operators (or rather, their classical counterparts), and are closely connected with certain components of the energy-momentum tensor $T^{\mu\nu}$ in (3.3). Much is known about the Kač-Moody algebra and its associated Virasoro algebra (an excellent review can be found in Ref. [50]), but the point I would like to emphasize here is that the L_m are generators of the group $\mathit{Diff}(S^1)$. It seems plausible that (5.4) and (5.5)—higher-dimensional generalizations of (5.6) and (5.7)—are in like manner associated with the group $\mathit{Diff}(\partial\Sigma)$, in particular $\mathit{Diff}(S^{d-2})$, of which much less is known. Of greatest interest is the case $d = 4$. Starting with the fact that $so(4) \cong so(3) \oplus so(3)$, it can be shown that (5.4) and (5.5) reduce to simply two commuting copies of the $d = 3$ case, (5.6) and (5.7), except that we must use the mode structure constants C_{MNP} of the spherical harmonics in place of the simpler $C_{mnp} = \delta_{m+n,p}$. So, at least from this point of view, understanding $\mathit{Diff}(S^2)$, and eventually the microstates responsible for the entropy of the four-dimensional black hole (in the approach considered here), does not seem that far out of reach.

6 Quantization, Microstates, and Black Hole Entropy

Before discussing quantization of the boundary theory let us consider the following. Boltzmann's formula tells us that the entropy, S, of a physical system is given by $\ln\Gamma$, where Γ is the number of microstates compatible with the macrostate the system is in. At the semiclassical level Γ is identified with (in a way I will not attempt to make precise here) the volume

of the phase space (or perhaps some subset of it). In Section 4 we found that the (reduced) phase space is the set of all maps from $\partial\Sigma$ (a constant Euclidean time slice of ∂M) into the Grassmann manifold $\mathrm{Gr}\,(2, d-2) = SO(d)/(SO(2)\times SO(d-2))$. For a Euclidean black hole with $M = R^2 \times S^{d-2}$ we have $\partial\Sigma = S^{d-2}$. (As argued in Section 2 we need only consider the inner boundary, i.e., we can take $\partial\Sigma$ to be a constant Euclidean time slice of just the boundary of the thickened bifurcation sphere.) Now clearly our whole analysis would go through unchanged if we instead considered $M = R^2 \times \mathcal{S}^{d-2}$, where $\mathcal{S}^{d-2}$ is any compact $(d-2)$-dimensional manifold without boundary, in which case $\partial\Sigma = \mathcal{S}^{d-2}$. So our phase space would consist of the set of all maps from $\mathcal{S}^{d-2}$ into $\mathrm{Gr}\,(2, d-2)$. To calculate the volume of this space (i.e., to determine Γ) would require a measure and careful regularization, which I will not attempt to do here. Instead I will give just a rough sketch of how one might proceed.

In place of the volume of the phase space let us think of Γ as the number of ways of embedding $\mathcal{S}^{d-2}$ into $\mathrm{Gr}\,(2, d-2)$, treated as a problem in combinatorics: Partition $\mathcal{S}^{d-2}$ into a number, α, of cells, and $\mathrm{Gr}\,(2, d-2)$ into β cells. There is a natural choice for α. Let A denote the 'area' (or volume) of $\mathcal{S}^{d-2}$ which, in the limit of infinitesimal thickening (of the excised bifurcation $(d-2)$-surface), is equal to the area of the bifurcation $(d-2)$-surface. It is natural to take $\alpha = A/\kappa$, where κ is (a numerical factor times) the Planck area. Since each cell of $\mathcal{S}^{d-2}$ can be embedded independently into any one of the cells of $\mathrm{Gr}\,(2, d-2)$ (ignoring smoothness considerations), we have $\Gamma = \beta^{\alpha}$. So we find that $\ln\Gamma$ is proportional to A/κ (in any dimension, d), the answer we want. Admittedly this is a simple-minded argument. Nevertheless, although a more sophisticated argument might determine the proportionality constant, I find it hard to imagine how it could give a qualitatively different result (e.g., $\ln\Gamma$ depending nonlinearly on A/κ, perhaps even in a way involving the dimension, d)—this point will be important later.

But this is not the end of the story. I think the intriguing part of it is this: It is well known that the homotopy classes of maps from $\mathcal{S}^{d-2}$ into a Grassmann manifold are intimately connected with the theory of characteristic classes, in particular the Euler number, χ, of $\mathcal{S}^{d-2}$. (A relevant specific example can be found in Ref. [51].) So it seems quite plausible that a careful semiclassical calculation of $\ln\Gamma$ will produce not only a factor proportional to A/κ, but will also involve χ. Finally, observing that χ is also the Euler number of M ($= R^2 \times \mathcal{S}^{d-2}$ in our case), the above considerations are quite suggestive of the generalized entropy formula $S = (\chi/8)A$ given in (0.1) [5].

Now let us move from these semiclassical considerations to a quantization proper of the boundary theory. Equation (5.4) displays the fundamental Poisson bracket relations (analogous to $\{p, q\} = 1$) that on quantization will yield an infinite-dimensional Hilbert space of states (the microstates). A finite-dimensional subspace of *physical* microstates is obtained by imposing the constraints (5.5) as quantum operators acting on these microstates. The dimension of this space is now the Γ to be used in Boltzmann's formula $S = \ln\Gamma$, and can be thought of as essentially the degeneracy of microstates satisfying the L_{P_0} 'mass-shell constraint.'

Unfortunately, I do not yet know how to carry out this quantization (and counting) program for cases $d > 3$. Backtracking somewhat, the quantization program can be described

as the task of finding irreducible unitary highest weight representations of the Lie algebra corresponding to the group of all maps from $\partial\Sigma$ into G ($= SO(d)$ in the Euclidean case), subject to certain constraints (the quantum analogue of the classical reduction from U to $\tilde{U}$—see (4.1) and (4.3)). Finding such representations (in numerous contexts) is currently an active area of research (see, e.g., Refs. [52, 53, 54, 55, 58, 56, 57]). Interest originated in the study of anomalous gauge theories with chiral fermions [52], and more recently has seen applications in p-branes and the study of non-perturbative effects in string theory [53, 54]. (Ref. [53] contains at least a partial list of the many important applications of this branch of representation theory to physics.) Unfortunately, at present there remain many unresolved issues. For instance, it is often difficult to prove the unitarity of attempted representations [54, 55]; and instead of a simple c-number central charge (see below) one encounters a number of possible operator-valued Schwinger terms depending in a complicated way on additional external fields [53,54,55,56,57]. We will see below, in the $d = 3$ case, that a certain c-number central charge plays a pivotal role in the physics of entropy. I expect that the operator-valued Schwinger terms occurring in higher dimensions will play an equally important role, but it is precisely these terms which are difficult to determine. Thus I will not attempt to pursue a quantization of the $d > 3$ cases here.

Fortunately the $d = 3$ case involves simply a loop group; much is known about quantizing the Euclidean version of equations (5.6) and (5.7) (see, e.g., Ref. [50]). The (quantum) Kač-Moody algebra, obtained by replacing the Poisson bracket with i times the commutator bracket, is

$$[J^A_m, J^B_n] = i f^{AB}{}_C J^C_{m+n} + km\delta^{AB}\delta_{m+n,0}\,. \tag{6.1}$$

The additional term on the far right is a central extension. The c-number central charge, k, is of purely quantum mechanical origin—I emphasize that our classical analysis gives us no information on the value of this number. Now given (6.1) and the quantum version of (5.7) an asymptotic formula for Γ is available [59]:

$$\ln\Gamma \sim F(k)\,\frac{\mathrm{Vol}\,(\partial\Sigma)}{\kappa} \tag{6.2}$$

in the limit $\mathrm{Vol}\,(\partial\Sigma)/\kappa \to \infty$. Again, strictly speaking, the right hand side of (6.2) is a sum over the disconnected components of $\partial\Sigma$. But as argued in Section 2, and mentioned again in the semiclassical discussion above, we need only consider the component of $\partial\Sigma$ which is a constant Euclidean time slice of the boundary of the thickened bifurcation sphere. So in the limit of infinitesimal thickening, $\mathrm{Vol}\,(\partial\Sigma) = A$, the 'area' (in this case length) of the bifurcation one-sphere of the Euclidean BTZ black hole. Thus, with κ a Planck scale unit of area, A/κ is large for any black hole much larger than the Planck scale, and our analysis says that in this case the physical entropy is given by

$$S = F(k)\,\frac{A}{\kappa}\,. \tag{6.3}$$

The function F of the central charge k can, in principle, be worked out using results in Ref. [59], but for the purposes of the remainder of this section we will not need to know its precise form.

At this point I can state three main observations emerging from the analysis. First, the boundary theory contained in $I^{(1)}[e]$ leads to physical microstates living on the boundary of a thickened bifurcation sphere which are *sufficiently rich in number* to provide a statistical mechanical account of the entropy of the $d = 3$ Euclidean black hole: our answer for the entropy is a (variable) numerical factor times the horizon area. Let us pause to ask whether we can expect a similar result to be obtained in any dimension, d. In other words, will this approach—at the quantum level—give an entropy proportional to the area in any dimension? In the $d = 3$ case the factor A/κ in (6.3) comes from the square root of the constrained value of L_0 in the mass-shell constraint in (5.7). The higher dimensional analogue of (5.7) is (5.5), which has exactly the same quadratic dependence on $\mathrm{Vol}\,(\partial\Sigma)/\kappa$. So the question is, will the number-theoretic counting arguments for cases $d > 3$ give a degeneracy of microstates depending on the *square root* of $(\mathrm{Vol}\,(\partial\Sigma)/\kappa)^2$, as it does in the $d = 3$ case? Unfortunately the answer is not at all obvious because the counting arguments depend intimately on the details of quantization.[10] And insofar as the quantization of the boundary theory for cases $d > 3$ is fraught with technical challenges (some of which were mentioned above), I do not yet know the answer. However, the rough semiclassical argument sketched at the beginning of this section provides hope that this question will be answered in the affirmative.

The second main observation concerns the numerical factor multiplying the area (more precisely, A/κ) in the entropy results. The rough semiclassical argument suggests that this factor will be related to the Euler number, χ, and hence is of topological origin. On the other hand, the $d = 3$ quantum calculation resulting in (6.3) shows that this factor ($F(k)$) depends on a central charge; since this central charge is not present in the classical or semiclassical analyses this result emphasizes how deeply quantum mechanical the phenomenon of black hole entropy really is. (Observe that the $\hbar$ in the Planck-scale factor κ also, of course, shows that black hole entropy is a quantum phenomenon, but it is independent of the *details* of quantization, being present already at the semiclassical level.) This is highly intriguing on several accounts. At the most basic level it represents a concrete realization of a suggestion by Carlip and Teitelboim that the numerical factor relating the entropy of a Euclidean black hole to the area of its bifurcation sphere might have its origin in microstates living on the boundary of a thickened bifurcation sphere [35]. $F(k)$ is precisely this numerical factor. At a deeper level this observation hints at a possible connection with recent results in quantum geometry and black hole entropy achieved in the framework of non-perturbative canonical quantum gravity [17,18]. The spectrum of the area operator in that approach is known to be discrete, and have a minimum eigenvalue of zero, with the next eigenvalue—the "area-gap"—depending on the topology of the surface in question. In this sense "quantum geometry 'knows' about topology" [18]. This is somewhat analogous to what appears to be happening here. If the semiclassical argument presented here is correct, then the purely quantum factor $F(k)$ must be proportional to the Euler number (more properly speaking—in this one-dimensional case—some topological invariant applicable for odd-dimensional surfaces): the quantum microstates on the thickened bifurcation surface 'know' about the topology of the bifurcation surface. More generally, are 'quantum anomalies' of a diffeomorphism algebra associated with topology?

[10] I am indebted to Steven Carlip for pointing this out to me.

Consider also the following. In the non-perturbative canonical quantum gravity approach to black hole microstates as described in Ref. [18] it is pointed out that, strictly speaking, at the classical level there are no independent boundary degrees of freedom; it is only at the quantum level that independent boundary degrees of freedom arise, which then account for the black hole entropy. This is apparently in marked contrast to what happens here, where there *are* independent boundary degrees of freedom beginning at the classical level. But is this apparent contrast really so? Although we begin with classical degrees of freedom U, I emphasize again that the central charge in (6.1), and hence $F(k)$ in the entropy formula (6.3), is of purely quantum mechanical origin—it has no classical analogue: again, black hole entropy is a deeply quantum phenomenon.

The third main observation concerns the *nature* of the result in (6.3). It is well known, by a number of arguments independent of the analysis presented here, that the entropy of a black hole is given by

$$S = \sigma \frac{A}{\kappa}\,, \tag{6.4}$$

where σ is a known numerical factor (e.g., 2π for nonextremal black holes in four dimensions [60]). Our analysis does not yield σ, but rather by *demanding* that (6.3) reproduce the correct result (6.4), a unique numerical value for the central charge k is thus determined. I believe that this is significant. It is known that the irreducible unitary highest weight representations of the Virasoro algebra are determined completely by two numbers. The first is the eigenvalue of L_0 (analogous to the eigenvalue of $|J|^2$ in a quantum mechanical treatment of angular momentum), and the second is the value of k [50]. In our case, the first is proportional to $(A/\kappa)^2$, and the second is determined by equating (6.3) with (6.4). In this way a particular representation of the Virasoro algebra is determined by the analysis. Furthermore, regarding representations of the Kač-Moody algebra, it is known that the vacuum states must form a representation of g, and I suspect that certain vacuum solutions of the boundary equations will play a role here, but this remains to be investigated.

The question of how to fix k has an analogue in the non-perturbative canonical quantum gravity approach to black hole microstates [17]. As mentioned above in the Introduction, there the entropy of a large non-rotating black hole is found to be proportional to A/κ divided by the Immirzi parameter, γ. An appropriate choice for γ then yields the Bekenstein-Hawking entropy—the same argument used in the previous paragraph to determine k. It is worth noting an additional parallel between these two variable parameters: on the one hand, different values of k correspond to unitarily inequivalent representations of the Virasoro algebra (and different spectra of the Virasoro operators); on the other hand, different values of γ correspond to unitarily inequivalent representations of the canonical commutation relations (and a different spectrum of the area operator in loop quantum gravity) [61, 62]. In Ref. [17] it is remarked that the ambiguity in choosing γ can be resolved only with the help of additional input (such as demanding agreement with the Bekenstein-Hawking entropy formula).[11]

[11] In some respects this argument might seem unsatisfactory: shouldn't a statistical mechanical analysis uniquely determine the entropy without recourse to thermodynamics? It is worth pointing out that an 'internal' mechanism might be available in our case to do just so. Work in progress indicates that the

Another point regarding the importance of the value of the central charge lies in a result known as the "quantum equivalence theorem" [50]. This theorem may have interesting consequences for the analysis here. As an illustration of this theorem, in a phenomenon he called non-Abelian bosonization, Witten argued that the "level 1" (read: a certain value of k) $SO(N)$ WZW theory is "quantum equivalent" to a theory of N massless free fermions [63]. This is surprising, given that the two theories look quite different, and note that the result depends crucially on the value of k. Now consider that, on the one hand, Carlip's analysis [7] involves a WZW boundary theory; on the other hand, it is not inconceivable that our boundary action, being of the form $U^{-1}dU$, with U an orthogonal matrix, could be cast into a Dirac form involving spinors. [Further support for this possibility comes from work by Baez *et al* on the quantum gravity Hamiltonian for manifolds with boundary, which hints at the existence of a boundary theory of Weyl spinors [16]. This work is in the context of the loop variables approach to quantum gravity, and given the receptiveness of the Goldberg action (and hence $I^{(1)}[e]$ in four dimensions) to an Ashtekar variables formulation [23, 25, 26], it is tempting to seek a connection between the work in [16] and what is done here.] So it might be possible to demonstrate "quantum equivalence" between the two approaches using ideas analogous to Witten's non-Abelian bosonization. Such a result would presumably hinge on the precise numerical value of k. In any case, although the $d = 3$ boundary theory here and the one in Ref. [7] look quite different, they both purport to describe the same physics, and certainly the mathematics used in the final stages of counting microstates is strikingly similar: these are strong indications that there exists some hidden connection between the two descriptions.[12]

Finally, we have already seen several hints that the approach discussed here might be related to the string theory approach to black hole microstates. As another, perhaps more direct hint, the discussion in Section 4.5 of Ref. [50] establishes the rather surprising result that the Virasoro operators associated with a Lie algebra g, and those associated with a subalgebra $h \subset g$, are "quantum equivalent" provided merely that the two algebras have the same central charge. This suggests it might be possible to show—again, depending crucially on the value of k—that the boundary theory here is "quantum equivalent" to a string moving

function $F(k)$ is peaked at a unique value of k. If correct, this would provide a very satisfying physical argument to fix k; namely, choose k such that the entropy is maximized.

[12] One additional remark can be made. The current algebra in Carlip's analysis [7] is an $so(2,1) \oplus so(2,1)$ version of (6.1), with a central charge I will denote as $\tilde{k}$. Unlike my k, $\tilde{k}$ depends on the cosmological constant, Λ. Carlip argues that in the semiclassical regime Λ is small, making $\tilde{k}$ large, which has the effect of 'Abelianizing' his current algebra, and thus simplifying the task of counting microstates. (Remark: although Carlip is making a large $\tilde{k}$ approximation his calculation does depend on the value of $\tilde{k}$, and in such a way that, of course, the final result for the entropy is independent of Λ.) Notice that this would seem to suggest that the non-Abelian nature of the current algebra is not important to the final result, in marked contrast to the analysis here, in which a large k approximation option does not make any sense, and the non-Abelian nature of (6.1) is crucial. Apparently k and $\tilde{k}$ play very different roles. A hint towards understanding the relationship between Carlip's approach and the one here comes from work by Bañados and Gomberoff [64] in which they argue that in fact the non-Abelian nature of the current algebra *does* play an important role in Carlip's analysis, except that this role is inadvertently hidden: if one makes a large $\tilde{k}$ approximation at an earlier stage of the calculation, rather than near the end, one does not get the correct result for the entropy. (I am indebted to Jack Gegenberg for bringing Ref. [64] to my attention, and to Steven Carlip for clarification of his analysis.)

on the maximal torus of the rotation group, G. (In this regard, observe that the isotropy subgroup, H, in the $d = 3$ and 4 cases is also, in fact, the maximal torus subgroup of G.)

Another question which I have not yet touched on concerns corrections to the Bekenstein-Hawking entropy formula. It seems that the proposed action $I^{(1)}[e]$ provides a clean and simple way to formulate this question. Let $\bar{I}^{(1)}[e] = I^{(1)}[e] - I^{(1)}[e_{\rm vac}]$ denote the regularized Euclidean action, where $e_{\rm vac}$ is a suitable vacuum spacetime solution, as discussed in Section 2. (Notice that $e_{\rm vac}$ consists of $\hat{e}_{\rm vac}$ as well as $U_{\rm vac}$.) The partition function is given by

$$Z = \sum_e [de] \, \exp \bar{I}^{(1)}[e] \,. \tag{6.5}$$

We then write the measure as a product, $[de] = [d\hat{e}]\,[dU]$, and use the split action given in (1.6), suitably regularized. As a first approximation to Z we retain only the classical piece $\hat{e}_{\rm cl}$ in the sum over $\hat{e}$, corresponding to a classical black hole 'macrostate' solution. In this approximation the partition function reduces to

$$Z \approx \exp(\bar{I}^{(1)}[\hat{e}_{\rm cl}]) \sum_U [dU] \, \exp \bar{I}_B[U; \hat{e}_{\rm cl}] \,. \tag{6.6}$$

The exponential outside the sum is responsible for the Bekenstein-Hawking entropy formula. The functional measure $[dU]$ is based on the finite-dimensional Haar measure of the frame rotation group, G; its "functional" aspect is associated with the manifold ∂M, in particular, the boundary of the thickened bifurcation sphere, which is *closed and compact*. The integrand in $I_B[U; \hat{e}_{\rm cl}]$ is proportional to $\mathrm{Tr}\,(U^{-1}dU \wedge \hat{E}_{\rm cl})$ (see (1.7)), and thus it is plausible that the sum on the right hand side of (6.6) involves a functional determinant of the form $\mathrm{Det}_{\partial M}(d_{\rm cl})$. Here $d_{\rm cl}$ denotes a linear differential operator on ∂M constructed out of the exterior derivative, d, and $\hat{E}_{\rm cl}$; notice that this operator contains information about the 'macrostate' (such as the black hole mass and angular momentum). Also, the boundary action $I_B[U; \hat{e}]$ is invariant under $U \rightarrow U_0 U$, U_0 a constant element of G on ∂M; this symmetry may play a role in the zero modes of d_{cl}. In any case, given its simple form, it seems likely that the entropy correction implied by (6.6) can probably be worked out *nonperturbatively*. Now, as emphasized by Carlip [14], "any quantum mechanical statement about black holes is necessarily a statement about *conditional* probabilities." This means the following. Suppose we have a complete theory of quantum gravity, and $|\, bh, \psi\rangle$ denotes an eigenstate in which bh (a certain classical black hole), and ψ (a certain set of values for all the other classical configuration space degrees of freedom of our hypothetical theory), are sharply defined. When we make a quantum mechanical statement about black holes we are restricted to transition amplitudes of the form $\langle\, bh, \psi' \,|\, bh, \psi\rangle$. It is precisely this restriction which is reflected in the approximation leading to (6.6), where $\hat{e}_{\rm cl}$ corresponds to bh, and U to ψ. To the extent that a formula for the entropy of a black hole makes sense only within this restriction, (6.6) would appear to contain *all* of the relevant quantum gravitational physics. The possibility that (6.6) can be solved nonperturbatively is thus quite exciting.

My final point, which I have already discussed in Section 2, but wish to properly clarify here, regards the question of just where do the microstates that account for black hole entropy live? Although the action $I^{(1)}[e]$ is completely general, and in particular can be analyzed in a spacetime of any signature, the main results presented here have been in the

context of Euclidean gravity, and so I will restrict my comments to this case. Assuming a Euclidean black hole topology $R^2 \times S^{d-2}$, there is certainly an $S^1 \times S^{d-2}$ boundary at infinity, and our analysis says that microstates necessarily live here. But their degeneracy (and hence the entropy associated with them) is infinite. However, as argued in Section 2, the suitable vacuum spacetime corresponding to this black hole has precisely the same entropy, and when subtracted, yields a physical black hole entropy equal to zero. This means that the microstates living on the boundary at infinity are not physical. (Such a regularization procedure is exactly analogous to what is usually done to calculate the Bekenstein-Hawking entropy starting from the action given in (2.1) [27,22,32,33,34].) But there is a subtlety here: as argued in Section 2, if we are to take $I^{(1)}[e]$ seriously as an action based on an orthonormal frame, rather than the metric, both the mathematics and physics strongly suggest that we excise the bifurcation sphere from M (ignoring for now any other set of points which might also have to be excised). This introduces a thickened bifurcation sphere whose $S^1 \times S^{d-2}$ boundary (the inner boundary) I will denote as ∂M_i. This is where the physically relevant microstates live. But if we apply this excision principle to the black hole we must be prepared to do the same to the corresponding vacuum spacetime. In the case $d = 4$, for example, this vacuum spacetime is topologically $R^3 \times S^1$ and, arguing again as in Section 2, we must excise the point set $\{0\} \times S^1$. 'Thickening' this point set introduces an $S^2 \times S^1$ boundary which we shall denote as $\partial M_i^{\text{vac}}$. Now choose any constant Euclidean time slice in $\partial M_i^{\text{vac}}$ and denote it as $\partial \Sigma_i^{\text{vac}}$. Topologically $\partial \Sigma_i^{\text{vac}} = S^2$, but *metrically* $\partial \Sigma_i^{\text{vac}}$ disappears in the limit of shrinking the 'thickened point set' down to $\{0\} \times S^1$: the boundary theory on $\partial M_i^{\text{vac}}$ is trivial, and the corresponding 'subtraction entropy' is thus zero. On the other hand, for the $d = 4$ black hole, $\partial M_i = S^1 \times S^2$, $\partial \Sigma_i = S^2$, and $\text{Vol}\,(\partial \Sigma_i)$ becomes the volume of the bifurcation sphere in the limit of shrinking the thickened bifurcation sphere down to $\{0\} \times S^2$, and this does *not* disappear. The boundary theory on ∂M_i is *not* trivial, and is responsible for the black hole entropy.

I argued in Section 2 that this approach to black hole microstates is fundamentally topological in nature, in that ∂M_i—the boundary on which the physically relevant microstates live—arises from a certain sensitivity of $I^{(1)}[e]$ to the topology of M. To clarify and strengthen this point recall that the entropy predicted by the boundary theory on ∂M_i is proportional to $\text{Vol}\,(\partial \Sigma_i)/\kappa$ (see (6.2)). This is somewhat peculiar. *Nowhere* in the analysis leading to (6.2) do we require a *metrical* property of the S^1 sector of ∂M_i ($= S^1 \times \partial \Sigma_i$), only a topological one: the S^1 comes from removing a single point ($\times S^{d-2}$); certainly there is a Euclidean time coordinate on this S^1, periodic with period equal to the inverse temperature, but *metrically* ∂M_i (but not $\partial \Sigma_i$) disappears in the limit. So the physically relevant microstates live on the boundary of a thickened bifurcation sphere, but there is no length scale associated with this thickening. It is not 'physics at the Planck scale' on a horizon interpreted as a 'tangible' boundary, but rather 'physics at no scale' on a manifold whose *raison d'être* is topological in nature.

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